MATH 243 – SECOND EXAM REVIEW

March 17, 2008

This list is not a complete list of skills you will need to demonstrate on the exam but outlines the major skills you will need. **Studying only from this sheet is not a good idea** as you will be expected to know all of the definitions and theoretical results presented during the lectures. Some of these items may not be on this list. Included below are also some verbal descriptions of how to solve some of the important problems for this material. See the lecture notes and text for examples of these methods.

- **Sections 16.1, 16.2, 16.3**

(1) If $f(x, y)$ is a continuous function over a region R and $f(x, y) \geq 0$ over R, then the volume of the solid under $f(x, y)$ and over R is

$$\text{Volume under } f(x, y) = \iint_R f(x, y)\, dA.$$

Compare this formula to the one in Section 16.7.

(2) If R is any region in the xy-plane, then the area of R is given by

$$\text{Area}(R) = \iint_R dA.$$

(3) For functions $f(x, y)$ and $g(x, y)$ and a constant c, we have

$$\iint_R (cf(x, y) + g(x, y))\, dA = c\iint_R f(x, y)\, dA + \iint_R g(x, y)\, dA.$$

(4) If a region R can be split into regions R_1 and R_2 with no overlap, then

$$\iint_R f(x, y)\, dA = \iint_{R_1} f(x, y)\, dA + \iint_{R_2} f(x, y)\, dA.$$

(5) Know how to identify Type I and Type II regions R in the xy-plane. (Type I have constant x boundaries and y bounded by functions of x. Type II is the opposite. It usually helps to **sketch your region** first when setting up an integral.)

(6) **Fubini's Theorem.** If $R = [a, b] \times [c, d]$ is a rectangle in the xy-plane, then

$$\iint_R f(x, y)\, dA = \int_a^b \int_c^d f(x, y)\, dy\, dx = \int_c^d \int_a^b f(x, y)\, dx\, dy.$$

The order of integration also does not matter if R is both Type I and Type II.

(7) If R is both Type I and II, know how to describe it in both manners. (Think about # 43 in Section 16.3, where choosing the correct order determines whether you can evaluate the integral or not.)

- **Section 16.4**
  (1) For polar coordinates (r, θ):
  $$x = r\cos(\theta) \qquad y = r\sin(\theta) \qquad r^2 = x^2 + y^2.$$
  (2) Know examples of regions that are better described with polar coordinates, e.g., circular disks, "washer"-shaped regions (disk with a hole), and "flower"-shaped regions (given by $r = a\cos(b\theta)$ or $r = a\sin(b\theta)$).
  (3) If $f(x, y)$ is continuous on a region $R = \{(r, \theta) \mid \alpha \le \theta \le \beta,\ h_1(\theta) \le r \le h_2(\theta)\}$, where $0 \le \beta - \alpha \le 2\pi$, then
  $$\iint\limits_R f(x, y)\, dA = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} f(r\cos(\theta), r\sin(\theta))\, r\, dr\, d\theta.$$
  Do not forget the extra r in the integral!
- **Section 16.5**
  (1) If a flat region R has density function $\rho(x, y)$, then the mass of R is given by
  $$m = \text{mass}(R) = \iint\limits_R \rho(x, y)\, dA.$$
  (2) The center of mass $(\overline{x}, \overline{y})$ is given by
  $$\overline{x} = \frac{1}{m} \iint\limits_R x\rho(x, y)\, dA \qquad \text{and} \qquad \overline{y} = \frac{1}{m} \iint\limits_R y\rho(x, y)\, dA,$$
  where m is the mass calculated above.
  (3) If the region R has electric charge density given by $\sigma(x, y)$, then the total charge on R is given by
  $$\text{charge}(R) = \iint\limits_R \sigma(x, y)\, dA.$$
- **Section 16.6**
  (1) The area of a surface $z = f(x, y)$ over a region R is given by
  $$\text{S.A.} = \iint\limits_R \sqrt{[f_x(x, y)]^2 + [f_y(x, y)]^2 + 1}\, dA$$
  if the partial derivatives $f_x(x, y)$ and $f_y(x, y)$ are continuous.
  (2) You may notice that most of these problems rely on switching to polar coordinates.
- **Section 16.7**
  (1) If $B = [a, b] \times [c, d] \times [r, s]$ is a box in xyz-space, then
  $$\iiint\limits_B f(x, y, z)\, dV = \int_r^s \int_c^d \int_a^b f(x, y, z)\, dx\, dy\, dz.$$
  Any of the 5 other orders are O.K. to use as well when the limits are constant.

(2) "Good" regions E for triple integration are Type I (where z is bounded by functions of x and y), Type II (where x is bounded by functions of y and z), and Type III (where y is bounded by functions of x and z). In the Type I case, for example,

$$\iiint_E f(x,y,z)\,dV = \iint_R \left(\int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\,dz \right) dA,$$

where R is a region in the xy-plane. After evaluating the interior integral, we have a double integral left to evaluate as in Sections 16.1 – 16.4.

(3) If a solid E is more than one of Types I, II, or III above, know how to write all possible descriptions and limits of integration. (See # 31–32 in Section 16.7.)

(4) If E is a solid region, then the volume of E is given by

$$\mathrm{Vol}(E) = \iiint_E dV.$$

If E has density function $\rho(x,y,z)$, then the mass of E is

$$\mathrm{mass}(E) = \iiint_E \rho(x,y,z)\,dV.$$

(5) It is usually best to sketch a 3D picture of the solid E involved in your problem. Try to find functions of x and y that describe the "floor" and "roof" of E. (The floor is often a constant $z = a$ plane.) Then try to determine the boundary region R for x and y. (This will often, but not always, be the floor of your solid. If the floor is at $z = a$ and the roof is $z = f(x,y)$, then solve $a = f(x,y)$ to find equations defining the floor of your solid.)

(6) Remember that it may help to convert to polar coordinates for these kinds of problems to describe the "floor" of your solid.

- **Section 16.8**

(1) You should review Section 13.7 for the basics about cylindrical and spherical coordinates. For cylindrical coordinates (r,θ,z), r measures the distance from the origin in the xy-plane, θ is the angle above the x-axis in the xy-plane, and z is z:

$$x = r\cos\theta \qquad y = r\sin\theta \qquad z = z.$$

For spherical coordinates, ρ ("rho") measures the radial distance from the origin in space, θ is the angle in the xy-plane above the x-axis, and ϕ ("phi") is the angle off of the z-axis (in general, $0 \le \phi \le \pi$), and

$$x = \rho\cos\theta\sin\phi \qquad y = \rho\sin\theta\sin\phi \qquad z = \rho\cos\phi.$$

(2) Review from p. 876 the formulas in cylindrical and spherical coordinates for various surfaces. These formulas will help you decide when to use a particular coordinate system. For instance, if you have two geometric objects that both have simple descriptions in spherical coordinates, then you should use those coordinates.

(3) Remember the extra factors ("Jacobians") that must be included when you change coordinates. If E is a solid, then

$$\begin{aligned}\iiint\limits_E f(x,y,z)\,dx\,dy\,dz &= \iiint\limits_E f(r\cos\theta, r\sin\theta, z) r\,dz\,dr\,d\theta \\ &= \iiint\limits_E f(\rho\cos\theta\sin\phi, \rho\sin\theta\sin\phi, \rho\cos\phi)\rho^2\sin\phi\,d\rho\,d\theta\,d\phi.\end{aligned}$$