MATH 243 – FIRST EXAM REVIEW

February 11, 2008

Below are outlined major skills you need for the exam. If you have not done so already, **review Chapter 13** since it contains many relevant concepts that I will assume you know already.

- **Section 15.1**
  (1) Know how to determine the domain and range of a function of 2 or 3 variables. For 2 variables, be able to sketch the domain. (When determining the domain, think about which possible inputs are not allowed: "What could go wrong?" For the range, decide which real numbers can be outputs of the function.)

  (2) Match the graph of a function with the formula. (It often helps to ask when does $f(x, y) = 0$.)

  (3) Be able to sketch a few level curves for a function $f(x, y)$ or match level curve drawings with formulas. (Each level curve consists of all points (x, y) in the plane that satisfy $f(x, y) = k$ for a particular k. These are also the curves you get by slicing the surface $z = f(x, y)$ horizontally.)

  (4) Use a level curve drawing to estimate values of $f(x, y)$.
- **Section 15.3**
  (1) Know how to find partial derivatives for a function of multiple (2 or more) variables, e.g., f_x, f_y, f_{xx}, f_{xy}, and f_{yy}. Know the various notations for these derivatives. (To find $f_x(x, y)$ think of y as a constant and x as the only variable.)

  (2) Understand that f_x and f_y measure the slopes of tangent lines running parallel to the x and y axes, respectively. See Figures 2–5 on p949.

  (3) Know that $f_{xy} = f_{yx}$. (Clairaut's Theorem)

  (4) Know how to use implicit differentiation to find $\frac{\partial z}{\partial x}$, for instance, when z is defined implicitly by a function $F(x, y, z) = 0$. (Apply $\frac{\partial}{\partial x}$ to $F(x, y, z) = 0$ and include $\frac{\partial z}{\partial x}$ after differentiating each function of z. Solve for $\frac{\partial z}{\partial x}$ afterward.)
- **Section 15.4**
  (1) Find the tangent plane at (x_0, y_0, z_0) to a surface $z = f(x, y)$. (The tangent plane is given by $z - f(x_0, y_0) = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$.)

  (2) Use a tangent plane formula to approximate the values of $z = f(x, y)$ near a point P when $f(x, y)$ is differentiable. (If you write the equation for the tangent plane at P as $z = L(x, y)$, then $f(x, y) \approx L(x, y)$ for points near P.)

  (3) Know that some functions are not differentiable, i.e., a tangent plane is not always a good approximation for a function, but functions with continuous first partial derivatives are differentiable.
- **Section 15.5**
  (1) Know the chain rule. You should be able to find $\frac{\partial f}{\partial r}$, $\frac{\partial f}{\partial s}$, and $\frac{\partial f}{\partial t}$ for $f(x, y, z)$ when x, y, and z are each functions of r, s, and t (and simpler cases too).

(2) Know how to find derivatives $\frac{dy}{dx}$ for $f(x, y) = 0$, and $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ for $F(x, y, z)$ using the chain rule technique. (Compare this method to the implicit method in Section 15.3.)

- **Section 15.6**

(1) Given a function f of 2 or 3 variables, know how to compute the directional derivative along a *unit* vector $\mathbf{u} = \langle a, b\rangle$ or $\mathbf{u} = \langle a, b, c\rangle$. (Need $\mathbf{u}$ to have length 1 first, i.e., $\sqrt{a^2 + b^2} = 1$ or $\sqrt{a^2 + b^2 + c^2} = 1$. Then $D_{\mathbf{u}}(f) = f_x a + f_y b$ or $D_{\mathbf{u}}(f) = f_x a + f_y b + f_z c$. In either event, $D_{\mathbf{u}}(f) = \nabla f \cdot \mathbf{u}$.)

(2) Know what $D_{\mathbf{u}}(f)$ means graphically. (It shows the rate of incline for f when you are standing at a point and facing the direction given by $\mathbf{u}$. Figure 3 on p977 is a good reference.)

(3) At a point (x, y) or (x, y, z) on a function $f(x, y)$ or $f(x, y, z)$, the largest possible value of ∇f is the length of the vector ∇f. This maximum value occurs along the direction given by the vector ∇f, i.e., the gradient points in the direction of the steepest incline at any point.

(4) An implicitly defined surface $F(x, y, z) = k$ has tangent plane

$$F_x(x_0, y_0, z_0)(x - x_0) + F_y(x_0, y_0, z_0)(y - y_0) + F_z(x_0, y_0, z_0)(z - z_0) = 0$$

at (x_0, y_0, z_0). (Compare this to Section 15.4.) This occurs because the gradient $\nabla F = \langle F_x, F_y, F_z\rangle$ is the normal vector to the tangent plane, i.e., for an implicitly given surface $F(x, y, z) = k$, the gradient points perpendicularly out from the surface at every point. (See Figures 9 and 10 on pp984–985.)

- **Section 15.7**

(1) Know what a local maximum or minimum (extremum) of a function is. Points where $f_x = f_y = 0$ or where either first partial is undefined are called critical points. Local extrema occur at critical points, but not all critical points are extrema.

(2) Know how to find the critical points of a function $f(x, y)$ and use the Second Derivative Test to help determine what kind of critical points you have (local max, local min, or saddle point).

- **Section 15.8**

(1) To find the max and min values of $f(x, y, z)$ subject to a constraint $g(x, y, z) = k$, find all solutions to the system of 4 equations in 4 unknowns given by

$$\begin{aligned} \nabla f(x, y, z) &= \lambda \nabla g(x, y, z) \\ g(x, y, z) &= k \end{aligned}$$

for x, y, z (and λ). Evaluate f at each solution to see which has the largest and smallest outputs.

(2) Solving these systems is a bit of an art. Look for ways to quickly eliminate a variable. It often (but not always) helps to try cases where one variable is assumed 0 and where it is assumed not 0. (This is often good when a variable is a common factor on both sides of one of the equations.) See pp1003–1004 for other tips for solving systems of equations.