

**Invertible Matrix Theorem.** Let  $A$  be an  $n \times n$  matrix. The following are equivalent.

- (a)  $A$  is invertible.
- (b)  $A$  is row equivalent to  $I_n$ .
- (c)  $A$  has  $n$  pivots.
- (d)  $A\mathbf{x} = \mathbf{0}$  only has the trivial solution.
- (e) The columns of  $A$  form a linearly independent set.
- (f) The linear transformation  $\mathbf{x} \mapsto A\mathbf{x}$  is one-to-one.
- (g)  $A\mathbf{x} = \mathbf{b}$  has at least one solution for every  $\mathbf{b}$  in  $\mathbb{R}^n$ .
- (h) The columns of  $A$  span  $\mathbb{R}^n$ .
- (i) The linear transformation  $\mathbf{x} \mapsto A\mathbf{x}$  is onto.
- (j)  $A$  is a product of elementary matrices.
- (k)  $A\mathbf{x} = \mathbf{b}$  has a unique solution for every  $\mathbf{b}$  in  $\mathbb{R}^n$ .
- (l) There is an  $n \times n$  matrix  $C$  such that  $CA = I_n$ .
- (m) There is an  $n \times n$  matrix  $D$  such that  $AD = I_n$ .
- (n)  $A^T$  is invertible.
- (o)  $\det A \neq 0$ .
- (p) The columns of  $A$  form a basis for  $\mathbb{R}^n$ .
- (q)  $\text{rank } A = n$ .
- (r) The null space of  $A$  is trivial.
- (s)  $\text{nullity } A = 0$ .
- (t) The rows of  $A$ , as vectors in  $\mathbb{R}^n$ , form a basis for  $\mathbb{R}^n$ .
- (u)  $\lambda = 0$  is not an eigenvalue for  $A$ .

Statements (a) – (n) are shown in Chapter 2.

Statement (o) is shown in Chapter 3.

Statements (p) – (t) are shown in Chapter 4.

Statement (u) is shown in Chapter 5.