MATH 140 – THIRD EXAM SOLUTIONS

November 21, 2008

NAME: _____

INSTRUCTIONS:

(1) SHOW ALL WORK.
(2) Do not begin until instructed to do so.
(3) You have 55 minutes to complete the exam.
(4) You may use a calculator.
(5) If you use a graph or table produced by a calculator to answer a problem, indicate what the calculator showed you.
(6) When an **exact** answer is specified, a calculator approximation is not acceptable.
(7) Include units when appropriate.

PROBLEM	POINTS	SCORE
1	12	
2	15	
3	15	
4	18	
5	20	
6	10	
7	10	
TOTAL	100	

1. The equation $5y^3 - 7x^2y + y^4 + x + 20 = 0$ implicitly defines a curve in the xy-plane.

(a) (8 points) Compute the implicit derivative $\dfrac{dy}{dx}$ for the curve.

Apply $\dfrac{d}{dx}$ to the equation to find:

$$15y^2\frac{dy}{dx} - 14xy - 7x^2\frac{dy}{dx} + 4y^3\frac{dy}{dx} + 1 = 0.$$

Solving for $\dfrac{dy}{dx}$ yields:

$$\frac{dy}{dx} = \frac{14xy - 1}{15y^2 - 7x^2 + 4y^3}.$$

(b) (4 points) Compute the tangent line to the curve at $(2, 1)$.

Evaluating $\dfrac{dy}{dx}$ at $x = 2$ and $y = 1$ yields a slope of -3. Thus, the line is

$$y = -3(x - 2) + 1.$$

2. Two runners start from the same location at the same time. The first runs due east at 12 ft/s while the second runs due north at 9 ft/s. Let x be the distance the first runner has run at time t, and let y be the distance the second runner has run at time t.

(a) (6 points) Sketch the situation at time t and find the distance between the two runners after 10 seconds?

The sketch should be a right triangle with horizontal side labeled as x and vertical side labeled as y. Label the hypotenuse s. Then $s^2 = x^2 + y^2$, and $x(10) = 120$ while $y(10) = 90$. Therefore,

$$s(10) = \sqrt{120^2 + 90^2} = 150.$$

(b) (9 points) Find the rate of change between the two runners after 10 seconds.

Apply $\dfrac{d}{dt}$ to $s^2 = x^2 + y^2$ to find

$$2s\frac{ds}{dt} = 2x\frac{dx}{dt} + 2\frac{dy}{dt}.$$

Solving for $\dfrac{ds}{dt}$ and substituting the information at 10 seconds gives:

$$\frac{ds}{dt} = 15 \text{ ft/s}.$$

3. Suppose that $g(x)$ is a function with $g(-4) = 220$ and $g'(x) = \dfrac{3}{x^2 - 10}$.
(a) (7 points) Use a linear approximation to estimate the value of $g(-4.02)$.
(b) (5 points) Determine if $g(x)$ is concave up or down at $x = -4$.
(c) (3 points) Is your estimate of $g(-4.02)$ an overestimate or underestimate? Explain.

The tangent line at $x = -4$ is $L_{-4}(x) = g'(-4)(x + 4) + g(-4)$, and so

$$L_{-4}(x) = \frac{1}{2}(x + 4) + 220.$$

Thus,

$$g(-4.02) \approx L_{-4}(-4.02) = \frac{1}{2}(-4.02 + 4) + 200 = 219.99.$$

For concavity, $g''(x) = -3(x^2 - 10)^{-2}(2x)$, and so $g''(-4) > 0$. Therefore, $g(x)$ is **concave up** at $x = -4$. As a result, the tangent line at $x = -4$ is below the curve and so the estimate above is an **underestimate**.

4. (18 points) For this problem use $f(x) = x^4 + 4x^3 - 10$. Use Calculus to find the following: **(a)** intervals where $f(x)$ is increasing, **(b)** local minima and maxima, **(c)** intervals where $f(x)$ is concave up, and **(b)** inflection points.

We compute $f'(x)$ and $f''(x)$ first. Since $f(x)$ is a polynomial, there will be no undefined points for any of the functions.

$$f'(x) = 4x^3 + 12x^2 = 4x^2(x + 3) \qquad f''(x) = 12x^2 + 24x = 12x(x + 2).$$

The critical points are then $x = 0$ and $x = -3$. The points where $f''(x) = 0$ are $x = 0$ and $x = -2$. Checking the number line for $f'(x)$ using the first derivative test, we can determine that $f(x)$ is increasing on $(-3, \infty)$ and that $x = -3$ is a local minimum. There is no local maximum. Checking the number line for $f''(x)$ shows that $f(x)$ is concave up on $(-\infty, -2) \cup (0, \infty)$, and so $x = 0$ and $x = -2$ are inflection points.

5. Consider the function $g(x) = (x+2)^2(x-8)^2 + 10$.

(a) (9 points) Using Calculus, find all critical points for $g(x)$. *Hints:* Use the product and rule and factor the result. All critical numbers are integers.

$$g'(x) = 2(x+2)(x-8)^2 + 2(x+2)^2(x-8) = 2(x+2)(x-8)(2x-6).$$

Setting $g'(x) = 0$ shows that the critical points are

$$x = -2, \quad x = 3, \quad x = 8.$$

(b) (6 points) Using Calculus, find the absolute minimum and maximum of $g(x)$ on the interval $[0, 7]$.

We need to check $g(x)$ at the endpoints and any critical points in the interval.
So,

$$g(0) = 266, \quad g(3) = 635, \quad g(7) = 91.$$

Thus the absolute maximum is 635 and the absolute minimum is 91.

(c) (5 points) Does there exist an input $x = c$ such that $g'(c) = -25$ in the interval $[0, 7]$? Explain why or why not briefly, including the name of an important theorem. (You need **not** find the value of c.)

We use the Mean Value Theorem:

$$\frac{g(7) - g(0)}{7 - 0} = \frac{91 - 266}{7} = -25.$$

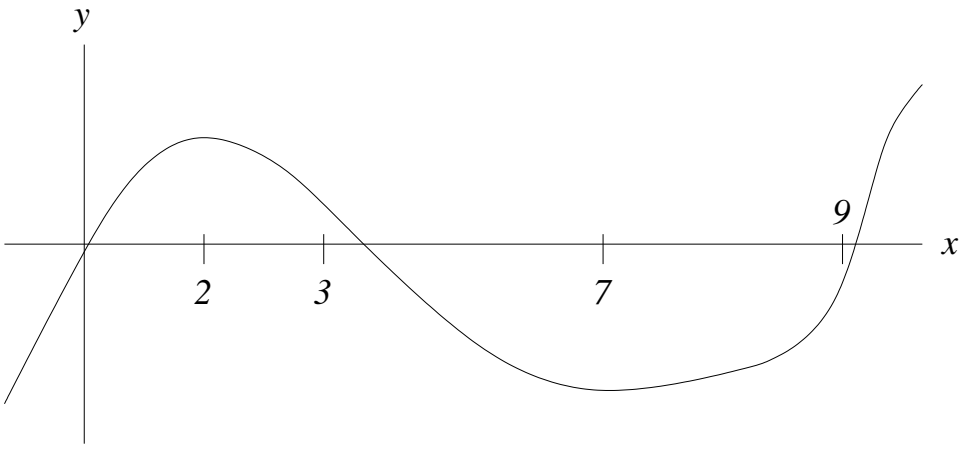
Thus, since $g(x)$ is continuous on $[0, 7]$ and differentiable on $(0, 7)$, the Mean Value Theorem states there is some c in $(0, 7)$ such that $g'(c) = -25$.

6. (10 points) Sketch one continuous function $f(x)$ that has the following properties:

$$f'(x) < 0 \text{ when } 2 < x < 7 \quad f'(x) > 0 \text{ when } x < 2 \text{ or } x > 7 \quad f'(2) = 0 \quad f'(7) = 0$$

$$f''(x) < 0 \text{ when } x < 3 \text{ or } x > 9 \quad f''(x) > 0 \text{ when } 3 < x < 9$$

Using the information above, we check for increasing and decreasing as well as concavity on the intervals $(-\infty, 2)$, $(2, 3)$, $(3, 7)$, $(7, 9)$, and $(9, \infty)$. A graph similar to below can then be sketched.



7. (10 points) Decide whether each statement is true or false and circle the correct answer.

(a) A function $f(x)$ defined on an interval (a, b) will never have an absolute maximum. **TRUE** **(FALSE)**

(b) A continuous function $g(x)$ defined on $[a, b]$ will always have an absolute minimum. **(TRUE)** **FALSE**

(c) If $f'(x) = g'(x)$, then $f(x) = g(x)$. **TRUE** **(FALSE)**

(d) If $h'(c) = 0$, then c is a local extremum. **TRUE** **(FALSE)**

(e) A continuous function is always differentiable. **TRUE** **(FALSE)**